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Holomorphic spectrum of twisted Dirac operators on compact Riemann surfaces

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Abstract

Given a Hermitian line bundle L with a harmonic connection over a compact Riemann surface (S, g) of constant curvature, we study the spectral geometry of the corresponding twisted Dirac operator $\mathcal{D}$. This problem is analyzed in terms of the natural holomorphic structures of the spinor bundles $\mathbb{E}^{\pm}$ defined by the Cauchy–Riemann operators associated with the spinorial connection. By means of two elliptic chains of line bundles obtained by twisting $\mathbb{E}^{\pm}$ with the powers of the canonical bundle K_S, we prove that there exists a certain subset $\mathrm{Spec}_{hol}(\mathcal{D})$ of the spectrum such that the eigensections associated with $\lambda \in \mathrm{Spec}_{hol}(\mathcal{D})$ are determined by the holomorphic sections of a certain line bundle of the elliptic chain. We give explicit expressions for the holomorphic spectrum and the multiplicities of the corresponding eigenvalues according to the genus p of S, showing that $\mathrm{Spec}_{hol}(\mathcal{D})$ does not depend on the spin structure and depends on the line bundle L only through its degree. This technique provides the whole spectrum of $\mathcal{D}$ for genus $p = 0$ and 1, whereas for genus $p > 1$ we obtain a finite number of eigenvalues.

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1. Introduction

Hitchin proved in [16] that there exists a one-to-one correspondence between the spin structures of a complex Hermitian manifold X and the holomorphic square roots $K_X^{\frac{1}{2}}$ of

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its canonical line bundle. If in addition X is Kähler, then the twisted Dirac operator $\mathcal{D}$ associated with a Hermitian holomorphic vector bundle E is completely determined by the Cauchy–Riemann operator $\bar{\partial}$ of $E \otimes K_X^{\frac{1}{2}}$ and we get explicitly $\mathcal{D} = \sqrt{2}(\bar{\partial}^* + \bar{\partial})$ (see for instance [4]). Therefore one may expect a close relationship between the holomorphic geometry of $E \otimes K_X^{\frac{1}{2}}$ and the spectral geometry of $\mathcal{D}$. In fact, by means of Hodge theory, one proves that the kernel of $\mathcal{D}$ on a compact Kähler spin manifold can be identified with the cohomology groups $H^\bullet\left(X, \mathcal{O}_X\left(E \otimes K_X^{\frac{1}{2}}\right)\right)$. This identification has been used by several authors to compute the harmonic spinors of certain Kähler manifolds, see for instance [1,5,8,16,19]. However, apart from the study of the kernel of the Dirac operator and some geometric estimates for the first eigenvalue [7,18,20], the holomorphic techniques have been used very little in the explicit computation of the spectrum of this operator.

One of the aims of this paper is to show that, under suitable conditions, it is possible to extend the above mentioned relationship to eigenvalues different from zero. In the present paper we restrict our attention to the particular case of a compact Riemann surface S endowed with a Riemannian metric of constant curvature and the twisting bundle is assumed to be a line bundle endowed with a harmonic connection. With these assumptions, we are able to prove that there exists a subset $\mathrm{Spec}_{hol}(\mathcal{D})$ of the spectrum of the twisted Dirac operator $\mathcal{D}$ which can be explicitly computed by means of holomorphic techniques. Moreover, for every $\lambda \in \mathrm{Spec}_{hol}(\mathcal{D})$, the corresponding eigensections can be identified with the holomorphic sections of a line bundle belonging to an elliptic chain defined over S. Due to these facts we call $\mathrm{Spec}_{hol}(\mathcal{D})$ the holomorphic spectrum of $\mathcal{D}$.

Let us explain now our method for analyzing $\mathrm{Spec}_{hol}(\mathcal{D})$ for a compact Riemann surface.

Any compact Riemannian surface (S, g) of genus p has 2^{2p} non-equivalent spin structures. If we fix one of them, then the spinor bundle is the $\mathbb{Z}_2$-graded vector bundle $\mathbb{S} = \Lambda^{0,\bullet}T^*(S) \otimes K_S^{\frac{1}{2}} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Given a Hermitian line bundle L endowed with a harmonic connection ∇, the associated twisted Clifford module is $\mathbb{E} = \mathbb{S} \otimes L = \mathbb{E}^+ \oplus \mathbb{E}^-$. The twisted Dirac operator is odd with respect to this $\mathbb{Z}_2$-gradation, therefore we write $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$. It is well known that the spectral resolution of $\mathcal{D}$ is completely determined by its kernel and the spectral resolution of $\mathcal{D}^-\mathcal{D}^+$ or $\mathcal{D}^+\mathcal{D}^-$.

We consider two chains of line bundles $\mathcal{C}^\bullet(\mathbb{E}^\pm) = \{K_S^q \otimes \mathbb{E}^\pm\}_{q\in\mathbb{Z}}$. The twisted spinorial connection on $\mathbb{E}^\pm$ and the Levi-Civita connection on K_S^q induce an integrable unitary connection $\nabla_\pm^q$ on each bundle $K_S^q \otimes \mathbb{E}^\pm$. The associated Cauchy–Riemann operators $\partial^{\nabla_\pm^q}$ and $\bar{\partial}^{\nabla_\pm^q}$ are elliptic and define morphisms between the differentiable sections of the bundles of each elliptic chain. For $q = 0$, one has $\mathcal{D}^+ = \sqrt{2}\,\bar{\partial}^{\nabla_+^0}$ and $\mathcal{D}^- = -\sqrt{2}\,\partial^{\nabla_-^0}$. On the elliptic chains we define the Laplacians $\Delta_\pm^q = (\partial^{\nabla_\pm^q})^*\partial^{\nabla_\pm^q}$ and $\Delta_q^\pm = (\bar{\partial}^{\nabla_\pm^q})^*\bar{\partial}^{\nabla_\pm^q}$. Therefore, $\mathcal{D}^-\mathcal{D}^+ = 2\,\Delta_0^+$, $\mathcal{D}^+\mathcal{D}^- = 2\,\Delta_-^0$ and the spectral resolution of $\mathcal{D}^2$ can be expressed in terms of the Laplacians of the elliptic chains.

If (S, g) is a Riemann surface of constant scalar curvature κ then the Kähler identities for $\nabla_\pm^q$ give the following fundamental commutation relations

$$\Delta_{-q}^{\pm}\partial^{\nabla_\pm^{-(q+1)}} - \partial^{\nabla_\pm^{-(q+1)}}\Delta_{-(q+1)}^{\pm} = \left[(q+1)\frac{\kappa}{2} \mp \frac{\kappa}{4} + B\right]\partial^{\nabla_\pm^{-(q+1)}}$$

$$\Delta_\pm^q\bar{\partial}^{\nabla_\pm^{(q+1)}} - \bar{\partial}^{\nabla_\pm^{(q+1)}}\Delta_\pm^{(q+1)} = \left[(q+1)\frac{\kappa}{2} \pm \frac{\kappa}{4} - B\right]\bar{\partial}^{\nabla_\pm^{(q+1)}}$$

where the curvature of ∇ is given by $iF^{\nabla} = B\,\omega$, ω is the Riemannian area element of S and B is constant since the connection ∇ of L is harmonic. These formulae are the key for giving explicit expressions for the eigenvalues of $\mathcal{D}$.

We point out that the elliptic chains $\mathcal{C}^{\bullet}(\mathbb{E}^{+})$ and $\mathcal{C}^{\bullet}(\mathbb{E}^{-})$ are equivalent and, in order to compute the spectral resolution of $\mathcal{D}^{-}\mathcal{D}^{+}$ (resp. $\mathcal{D}^{+}\mathcal{D}^{-}$), we make use only of $\mathcal{C}^{\bullet}(\mathbb{E}^{+})$ (resp. $\mathcal{C}^{\bullet}(\mathbb{E}^{-})$).

If $\deg \mathbb{E}^{+} > 0$ then any non-vanishing holomorphic section $s^{-q} \in H^0(S, K_S^{-q} \otimes \mathbb{E}^{+})$ with $q \leq 0$ allows us to define an eigensection of $\mathcal{D}^{-}\mathcal{D}^{+}$ by

$$s_q = \partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} s^{-q}.$$

Although the case $\deg \mathbb{E}^{+} < 0$ can be dealt with in a similar way by considering the space of antiholomorphic sections of the bundle $K_S^{q} \otimes \mathbb{E}^{+}$, there is a non-trivial shift in the spectrum, with respect to the case of positive degree. This is due to the equality $\Delta_0^{+} = \Delta_{+}^{0} + (\frac{\kappa}{4} - B)$ which appears when one computes the spectrum of $\mathcal{D}^{-}\mathcal{D}^{+}$. The $\deg \mathbb{E}^{+} = 0$ case is analyzed in a similar way to the positive degree case, except for genus $p = 0$ where a different analysis is required due to the fact that every holomorphic section vanishes.

Summarizing, if the spaces $H^0(S, K_S^{-q} \otimes \mathbb{E}^{+})$, $H^0(S, K_S^{q} \otimes (\mathbb{E}^{+})^{-1})$ are not trivial then the preceding method allows us to obtain eigensections of $\mathcal{D}$ and their eigenvalues, which form the holomorphic spectrum $\mathrm{Spec}_{hol}(\mathcal{D})$, can be explicitly computed by using the commutation relations. The multiplicity of each eigenvalue is given by the Riemann–Roch theorem. As we shall see, in some cases we can find with this technique the whole spectrum of the twisted Dirac operator. Moreover, since we obtain an explicit expression for the eigenvalues, it is easy to check the dependence of the spectrum on the spin structure and on the twisting line bundle. Our method also allows us to prove the existence of harmonic spinors and to compute its dimension.

If the genus of the Riemann surface is $p = 0$ we prove that $\mathrm{Spec}_{hol}(\mathcal{D})$ is the whole spectrum. For the twisted Dirac operator with the Fubini–Study metric of unit volume, the eigenvalues and their multiplicities coincide with those obtained by Bär and Schopka in [6] by means of different techniques. In the untwisted case we recover the well-known results concerning the spectrum of the Dirac operator on S^2 (see [29,9]). However our method has the advantage of giving explicit expressions for the eigensections in terms of complex homogeneous polynomials of degree $2q + |\deg L - 1|$. It also clarifies the shift in the spectrum for line bundles of negative degree.

For genus $p = 1$, the case of the untwisted Dirac operator is covered by the well-known results of Friedrich [12] on the spectrum of the Dirac operator on flat tori and its dependence on the spin structure. Similar results have been obtained very recently by Miatello and Podesta for the Dirac operator twisted by flat line bundles on compact Bieberbach manifolds [24]. These authors make essential use of the spectrum of the Laplacian on flat tori. In the present paper, we generalize their results to the case of the Dirac operator twisted by a line bundle L of arbitrary degree on any Riemann surface S of genus $p = 1$ endowed with a flat Riemannian metric. Let us point out that when the degree of L is $k \neq 0$ then $\mathrm{Spec}_{hol}(\mathcal{D})$ coincides with $\mathrm{Spec}(\mathcal{D})$. Moreover, we prove that it does not depend on the chosen spin structure and it is constant over the component of the Picard group $\mathrm{Pic}^k(S)$ which parametrizes line bundles of degree k. In this case the eigensections can be expressed in terms of theta functions with characteristic. As we have mentioned above, the $\deg L = 0$ case cannot be studied with the holomorphic techniques since the spaces of holomorphic sections $H^0(S, K_S^{-q} \otimes \mathbb{E}^{+})$ vanish unless L is the trivial holomorphic line bundle. For the sake of completeness, we have studied this case by means of a method which is close to the ideas of Friedrich, since we make use of the spectrum of the Laplacian, but different in spirit since we use the Picard group $\mathrm{Pic}^0(S)$ to parametrize both the holomorphic line bundles of degree zero and the gauge equivalence classes of flat unitary connections on the trivial C^∞ line bundle. The results that we obtain for the spectrum show, as one could expect, an explicit dependence on the spin structure. One also checks that there are harmonic spinors only for the trivial spin structure. On the other hand, the computation of the multiplicity of the eigenvalues reduces to a non-trivial arithmetic problem. If we fix a lattice Λ such that $S \simeq \mathbb{C}/\Lambda$, then it is possible to give geometrical estimates for the first eigenvalue by means of simple graphical methods which involve the Dirichlet fundamental domain of Λ, also known as the Voronoi polygon of Λ (see for instance [25]). In this way we are able to give a simple geometric interpretation to the first Vafa–Witten invariant [30,2] (this is an upper bound for the absolute value of the first eigenvalue which does not depend either on the line bundle L or on the spin structure) as the minimum of the distances from the origin to the vertices of the Voronoi polygon associated with a reduced basis of Λ.

If the genus is $p > 1$ then $\mathrm{Spec}_{hol}(\mathcal{D})$ is a finite subset of the spectrum $\mathrm{Spec}(\mathcal{D})$. In this case, in order for our method to work, the weight of the line bundle $k(L) = \frac{\deg L}{\deg K_S}$ has to be greater than $\frac{1}{2}$ or less that $-\frac{3}{2}$. This excludes the case $\deg L = 0$, which covers the untwisted case. Although the computation of the whole spectrum continues to be an open problem, to the best of our knowledge the eigenvalues for the twisted Dirac operator that we obtain in this paper were not previously known in the literature. One can prove that the holomorphic spectrum $\mathrm{Spec}_{hol}(\mathcal{D})$ consists of the lowest eigenvalues [26]. On the other hand we show that $\mathrm{Spec}_{hol}(\mathcal{D})$ does not depend on the spin structure and the eigensections correspond to automorphic forms of weight $k(L)$ (for instance see [14]).

The paper is organized as follows. In Section 2 we establish our notation and we recall some known facts concerning twisted Dirac operators on compact Riemann surfaces. In Section 3 we introduce the elliptic chains associated with the twisted Dirac operator and we prove the fundamental commutation relations between Cauchy–Riemann operators and the spinorial Laplacians. In Section 4 we prove that the spectral resolution of $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$ is completely determined by that of $\mathcal{D}^-\mathcal{D}^+$. The proof is based on a natural quaternionic structure carried by the spaces of eigensections of $\mathcal{D}^2$ and which is defined in a natural way by the twisted Dirac operator $\mathcal{D}$ and the $\mathbb{Z}_2$-gradation of the spinor bundles. Finally, in Section 5 we study the spectrum of the twisted Dirac operator corresponding to harmonic connections on any compact Riemann surfaces of constant curvature, the results are given in separate subsections according to the genus.

2. Dirac operators on Riemann surfaces

In order to establish our notation we begin by briefly recalling some facts concerning Dirac operators on Riemann surfaces.

Let S be a compact Riemann surface endowed with a Riemannian metric g of constant curvature. It is well known (for instance see [3]) that S is a spin manifold and a choice of a spin structure on it is given by a square root of its canonical bundle $K_S = \Lambda^{1,0}T^*(S)$, i.e. a complex line bundle $\mathcal{L}$ over S such that $\mathcal{L} \otimes \mathcal{L} \simeq K_S$. Traditionally such a line bundle is denoted by $K_S^{\frac{1}{2}}$ although the square root is not unique. In fact, if p denotes the genus of S, then there are 2^{2p} non-equivalent spin structures on S.

The bundle of complex spinors associated with a spin structure $K_S^{\frac{1}{2}}$ is the Hermitian bundle

$$\mathbb{S} = \Lambda^{0,\bullet}T^*(S) \otimes K_S^{\frac{1}{2}}$$

where $\Lambda^{0,\bullet}T^*(S)$ is the vector bundle of complex forms of type $(0,q)$ with arbitrary q. This vector bundle has a $\mathbb{Z}_2$-gradation $\mathbb{S}=\mathbb{S}^+\oplus\mathbb{S}^-$ given by

$$\mathbb{S}^+ = \Lambda^{0,0}T^*(S)\otimes K_S^{\frac{1}{2}} \simeq K_S^{\frac{1}{2}}$$
$$\mathbb{S}^- = \Lambda^{0,1}T^*(S)\otimes K_S^{\frac{1}{2}} \simeq K_S^{-\frac{1}{2}}$$

where the second isomorphism follows from the natural identification of $\Lambda^{0,1}T^*(S)$ with the anti-canonical line bundle K_S^{-1} established by the metric. We denote by $\nabla_{\mathbb{S}}$ the spinorial connection of $\mathbb{S}$ induced by the Levi-Civita connection, see [22].

Now let us consider a Hermitian line bundle L on S endowed with a unitary harmonic connection ∇. Its curvature $F^\nabla\in\Omega^{1,1}(S)$ can be expressed as

$$iF^\nabla = B\,\omega$$

where ω is the Riemannian area element of S and $B\in\mathbb{R}$.

The twisted Clifford module $\mathbb{E}=\mathbb{S}\otimes L$ has a natural decomposition $\mathbb{E}=\mathbb{E}^+\oplus\mathbb{E}^-$ with

$$\mathbb{E}^+ = \mathbb{S}^+\otimes L \simeq K_S^{\frac{1}{2}}\otimes L$$
$$\mathbb{E}^- = \mathbb{S}^-\otimes L \simeq K_S^{-\frac{1}{2}}\otimes L.$$

The product connection $\nabla_{\mathbb{E}}=\nabla_{\mathbb{S}}\otimes 1+1\otimes\nabla$ on $\mathbb{E}$ is a Hermitian Clifford connection which determines the twisted Dirac operator

$$\mathcal{D}:\Omega^0(S,\mathbb{E})\to\Omega^0(S,\mathbb{E})$$

where $\Omega^0(S,\mathbb{E})$ is the space of C^∞ sections of $\mathbb{E}$. We denote by $\mathcal{D}^\pm$ the restriction of $\mathcal{D}$ to the spaces $\Omega^0(S,\mathbb{E}^\pm)$

$$\mathcal{D}^+:\Omega^0(S,\mathbb{E}^+)\longrightarrow\Omega^0(S,\mathbb{E}^-)$$
$$\mathcal{D}^-:\Omega^0(S,\mathbb{E}^-)\longrightarrow\Omega^0(S,\mathbb{E}^+).$$

If $c_{\mathbb{E}}:Cl(T^*S)\otimes\mathbb{C}\to\mathrm{End}(\mathbb{E})$ is the twisted spinor representation of the complex Clifford algebra of (S,g) on $\mathbb{E}$, then one has the Lichnerowicz–Weitzenböck formula

$$\mathcal{D}^2=\nabla_{\mathbb{E}}^*\nabla_{\mathbb{E}}+c_{\mathbb{E}}(F^{\nabla_{\mathbb{E}}})=\Delta_{\mathbb{E}}+\frac{\kappa}{4}\,\mathrm{Id}_{\mathbb{E}}+c_{\mathbb{E}}(F^\nabla)$$

where κ is the scalar curvature of (S,g) and $\Delta_{\mathbb{E}}=\nabla_{\mathbb{E}}^*\nabla_{\mathbb{E}}$ is the Bochner Laplacian associated with $\nabla_{\mathbb{E}}$. With respect to the decomposition $\mathbb{E}=\mathbb{E}^+\oplus\mathbb{E}^-$ one has $c_{\mathbb{E}}(F^\nabla)=-B\,\mathrm{Id}_{\mathbb{E}^+}\oplus B\,\mathrm{Id}_{\mathbb{E}^-}$, hence

$$\mathcal{D}^2=\Delta_{\mathbb{E}}+\left[\left(\frac{\kappa}{4}-B\right)\mathrm{Id}_{\mathbb{E}^+}\oplus\left(\frac{\kappa}{4}+B\right)\mathrm{Id}_{\mathbb{E}^-}\right].$$

Therefore, in matrix notation, we can write

$$\mathcal{D}^2=\begin{pmatrix}\mathcal{D}^-\mathcal{D}^+ & 0\\ 0 & \mathcal{D}^+\mathcal{D}^-\end{pmatrix} \tag{1}$$

with

$$\mathcal{D}^-\mathcal{D}^+=\Delta_{\mathbb{E}^+}+\left[\frac{\kappa}{4}-B\right]\mathrm{Id}_{\mathbb{E}^+}$$

$$\mathcal{D}^+\mathcal{D}^- = \Delta_{\mathbb{E}^-} + \left[\frac{\kappa}{4} + B\right]\mathrm{Id}_{\mathbb{E}^-}$$

where $\Delta_{\mathbb{E}^\pm}$ is the Bochner Laplacian of the connection $\nabla_{\mathbb{E}^\pm}$ defined on $\mathbb{E}^\pm$ by the decomposition $\nabla_{\mathbb{E}} = \nabla_{\mathbb{E}^+} \oplus \nabla_{\mathbb{E}^-}$.

The vector bundle $\mathbb{E}$ has a natural holomorphic structure determined by the Cauchy–Riemann operator $\bar\partial^{\nabla_{\mathbb{E}}}$ associated with the Hermitian connection $\nabla_{\mathbb{E}}$. If $\nabla_{\mathbb{E}} = \partial^{\nabla_{\mathbb{E}}} + \bar\partial^{\nabla_{\mathbb{E}}}$ is the decomposition of $\nabla_{\mathbb{E}}$ in terms of the Cauchy–Riemann operators, we have

$$\begin{aligned}\partial^{\nabla_{\mathbb{E}}} &: \Omega^{p,q}(S,\mathbb{E}) \longrightarrow \Omega^{p+1,q}(S,\mathbb{E})\\ \bar\partial^{\nabla_{\mathbb{E}}} &: \Omega^{p,q}(S,\mathbb{E}) \longrightarrow \Omega^{p,q+1}(S,\mathbb{E}).\end{aligned}$$

The Kähler identities lead to

$$\begin{aligned}\Delta_{\mathbb{E}} &= 2(\bar\partial^{\nabla_{\mathbb{E}}})^*\,\bar\partial^{\nabla_{\mathbb{E}}} + i\Lambda F^{\nabla_{\mathbb{E}}}\\ \Delta_{\mathbb{E}} &= 2(\partial^{\nabla_{\mathbb{E}}})^*\,\partial^{\nabla_{\mathbb{E}}} - i\Lambda F^{\nabla_{\mathbb{E}}}\end{aligned}$$

where Λ denotes the contraction by the Kähler form. In particular one has

$$i\Lambda F^{\nabla_{\mathbb{E}}} = \begin{pmatrix} -\frac{\kappa}{4} + B & 0 \\ 0 & \frac{\kappa}{4} + B \end{pmatrix}$$

and thus

$$\begin{aligned}\Delta_{\mathbb{E}^\pm} &= 2\,(\bar\partial^{\nabla_{\mathbb{E}^\pm}})^*\,\bar\partial^{\nabla_{\mathbb{E}^\pm}} + \left[B \mp \frac{\kappa}{4}\right]\mathrm{Id}_{\mathbb{E}^\pm}\\ \Delta_{\mathbb{E}^\pm} &= 2\,(\partial^{\nabla_{\mathbb{E}^\pm}})^*\,\partial^{\nabla_{\mathbb{E}^\pm}} - \left[B \mp \frac{\kappa}{4}\right]\mathrm{Id}_{\mathbb{E}^\pm}.\end{aligned}$$

Therefore we obtain the

Lemma 2.1. *On $\Omega^0(S,\mathbb{E}^\pm)$ we have the equality*

$$(\partial^{\nabla_{\mathbb{E}^\pm}})^*\,\partial^{\nabla_{\mathbb{E}^\pm}} - (\bar\partial^{\nabla_{\mathbb{E}^\pm}})^*\,\bar\partial^{\nabla_{\mathbb{E}^\pm}} = \left[B \mp \frac{\kappa}{4}\right]\mathrm{Id}_{\mathbb{E}^\pm}.$$

Finally, the components of the square of the Dirac operator can be written in terms of the Cauchy–Riemann operators of the holomorphic structure of $\mathbb{E}^\pm$ as

$$\mathcal{D}^-\mathcal{D}^+ = 2\,(\bar\partial^{\nabla_{\mathbb{E}^+}})^*\,\bar\partial^{\nabla_{\mathbb{E}^+}} = 2\,(\partial^{\nabla_{\mathbb{E}^+}})^*\,\partial^{\nabla_{\mathbb{E}^+}} + 2\left[\frac{\kappa}{4} - B\right] \tag{2}$$

$$\mathcal{D}^+\mathcal{D}^- = 2\,(\partial^{\nabla_{\mathbb{E}^-}})^*\,\partial^{\nabla_{\mathbb{E}^-}} = 2\,(\bar\partial^{\nabla_{\mathbb{E}^-}})^*\,\bar\partial^{\nabla_{\mathbb{E}^-}} + 2\left[\frac{\kappa}{4} + B\right]. \tag{3}$$

Since $\mathbb{E}^+ = K_S^{\frac{1}{2}} \otimes L$, it is well known (see for instance [4]) that the Dirac operator itself can be expressed as

$$\mathcal{D} = \sqrt{2}\,[\bar\partial^{\nabla_{\mathbb{E}^+}} + (\bar\partial^{\nabla_{\mathbb{E}^+}})^*].$$

Moreover, since S is a one dimensional complex manifold, we have $\mathcal{D}^+ = \sqrt{2}\,\bar\partial^{\nabla_{\mathbb{E}^+}}$, $\mathcal{D}^- = \sqrt{2}\,(\bar\partial^{\nabla_{\mathbb{E}^+}})^*$ and the Kähler identities allow us to conclude that

Lemma 2.2. *On $\Omega^{0,1}(\mathbb{E}^\pm)$ we have the identity*

$$\bar\partial^{\nabla_{\mathbb{E}^\pm}}(\bar\partial^{\nabla_{\mathbb{E}^\pm}})^* = (\partial^{\nabla_{\mathbb{E}^\pm}})^*\partial^{\nabla_{\mathbb{E}^\pm}}.$$

3. Elliptic chains associated with the twisted Dirac operator on a compact Riemann surface

The relationship between Dirac and Cauchy–Riemann operators established in Section 2 allows us to analyze the spectral geometry of the operators $\mathcal{D}^-\mathcal{D}^+$ and $\mathcal{D}^+\mathcal{D}^-$ on a compact surface S by means of a chain of holomorphic line bundles endowed with elliptic differential operators between them.

Definition 3.1. We call $\mathcal{C}^\bullet(\mathbb{E}^\pm) = \{\mathcal{C}^q(\mathbb{E}^\pm) = K_S^q \otimes \mathbb{E}^\pm\}_{q\in\mathbb{Z}}$ the chains of line bundles associated with the Hermitian line bundles $\mathbb{E}^\pm$ and the unitary connections $\nabla_{\mathbb{E}^\pm}$.

We denote by $\nabla_\pm^q$ the connection on $K_S^q \otimes \mathbb{E}^\pm$ defined by twisting $\nabla_{\mathbb{E}^\pm}$ with the connection of K_S^q induced by the Levi-Civita connection. The Cauchy–Riemann operators of $\nabla_\pm^q$ give us the morphisms

$$\partial^{\nabla_\pm^q} : \Omega^0(S, K_S^q \otimes \mathbb{E}^\pm) \longrightarrow \Omega^{1,0}(S, K_S^q \otimes \mathbb{E}^\pm)$$
$$\bar\partial^{\nabla_\pm^q} : \Omega^0(S, K_S^q \otimes \mathbb{E}^\pm) \longrightarrow \Omega^{0,1}(S, K_S^q \otimes \mathbb{E}^\pm).$$

Notice that $K_S^q \otimes \mathbb{E}^\pm = K_S^{q\pm1} \otimes \mathbb{E}^\mp$, thus $\nabla_\pm^q = \nabla_\mp^{q\mp1}$ which implies

$$\partial^{\nabla_\pm^q} = \partial^{\nabla_\mp^{q\pm1}}$$
$$\bar\partial^{\nabla_\pm^q} = \bar\partial^{\nabla_\mp^{q\pm1}}.$$

In what follows we use these identifications without any further notice.

On the other hand the tensor product and the polarity of the Riemannian metric g provide the following C^∞ identifications

$$\Omega^{1,0}(S, K_S^q \otimes \mathbb{E}^\pm) \simeq \Omega^0(S, K_S^{q+1} \otimes \mathbb{E}^\pm)$$
$$\Omega^{0,1}(S, K_S^q \otimes \mathbb{E}^\pm) \simeq \Omega^0(S, K_S^{q-1} \otimes \mathbb{E}^\pm).$$

We continue to denote by the same symbols the Cauchy–Riemann operators after composing them with the above isomorphisms. Therefore we have the following differential operators between the elements of the chains $\mathcal{C}^\bullet(\mathbb{E}^\pm)$

$$\partial^{\nabla_\pm^q} : \Omega^0(S, K_S^q \otimes \mathbb{E}^\pm) \longrightarrow \Omega^0(S, K_S^{q+1} \otimes \mathbb{E}^\pm)$$
$$\bar\partial^{\nabla_\pm^q} : \Omega^0(S, K_S^q \otimes \mathbb{E}^\pm) \longrightarrow \Omega^0(S, K_S^{q-1} \otimes \mathbb{E}^\pm).$$

The following is well known

Proposition 3.2. *The differential operators* $\partial^{\nabla_\pm^q}$ *and* $\bar\partial^{\nabla_\pm^q}$ *are elliptic and*

$$(\partial^{\nabla_\pm^q})^* = -\bar\partial^{\nabla_\pm^{q+1}}$$
$$(\bar\partial^{\nabla_\pm^q})^* = -\partial^{\nabla_\pm^{q-1}}.$$

Remark 3.3. For $q = 0$ we have $\nabla_\pm^0 = \nabla_{\mathbb{E}^\pm}$, thus $\mathcal{D}^+ = \sqrt{2}\,\bar\partial^{\nabla_+^0} = \sqrt{2}\,\bar\partial^{\nabla_-^{-1}}$ and $\mathcal{D}^- = \sqrt{2}\,(\bar\partial^{\nabla_+^0})^* = -\sqrt{2}\,\partial^{\nabla_-^0} = -\sqrt{2}\,\partial^{\nabla_+^{-1}}$. Hence

$$\mathcal{D}^-\mathcal{D}^+ = -2\,\partial^{\nabla_-^0}\bar\partial^{\nabla_+^0} = -2\,\partial^{\nabla_+^{-1}}\bar\partial^{\nabla_+^0}$$
$$\mathcal{D}^+\mathcal{D}^- = -2\,\bar\partial^{\nabla_+^0}\partial^{\nabla_-^0} = -2\,\bar\partial^{\nabla_-^1}\partial^{\nabla_-^0}.$$

Definition 3.4. We call $(\mathcal{C}^\bullet(\mathbb{E}^\pm), \partial^{\nabla^\bullet_\pm}, \bar\partial^{\nabla^\bullet_\pm})$ the elliptic chains associated with the Hermitian line bundles $\mathbb{E}^\pm$ and the unitary connections $\nabla_{\mathbb{E}^\pm}$.

Proposition 3.5. *If κ is the scalar curvature of (S, g), one has*

$$\partial^{\nabla^{q-1}_\pm}\bar\partial^{\nabla^q_\pm} - \bar\partial^{\nabla^{q+1}_\pm}\partial^{\nabla^q_\pm} = -q\frac{\kappa}{2} \mp \frac{\kappa}{4} + B.$$

Proof. The line bundle $K_S^q \otimes \mathbb{E}^\pm$ of the elliptic chain is endowed with the connection $\nabla^q_\pm = \nabla_{K_S^q} \otimes 1 + 1 \otimes \nabla_{\mathbb{E}^\pm}$ and therefore its curvature is

$$F^{\nabla^q_\pm} = F^{\nabla_{K_S^q}} + F^{\nabla_{\mathbb{E}^\pm}} = qF^{\nabla_{K_S}} + F^{\nabla_{\mathbb{E}^\pm}}.$$

Since $i\,F^{\nabla_{K_S}} = -\frac{\kappa}{2}\,\omega$ and $i\,F^{\nabla_{\mathbb{E}^\pm}} = \left(B \mp \frac{\kappa}{4}\right)\omega$, we get $i\Lambda F^{\nabla_{K_S^q}} = -q\,\frac{\kappa}{2} \mp \frac{\kappa}{4} + B$. Now using ideas similar to those in [26, Proposition 9] and Lemma 2.1, one proves

$$(\partial^{\nabla^q_\pm})^*\partial^{\nabla^q_\pm} - (\bar\partial^{\nabla^q_\pm})^*\bar\partial^{\nabla^q_\pm} = -q\frac{\kappa}{2} \mp \frac{\kappa}{4} + B. \quad \square$$

Definition 3.6. On the elliptic chains $(\mathcal{C}^\bullet(\mathbb{E}^\pm), \partial^{\nabla^\bullet_\pm}, \bar\partial^{\nabla^\bullet_\pm})$ we define, for any $q \in \mathbb{Z}$, the Laplacians

$$\Delta^q_\pm = (\partial^{\nabla^q_\pm})^*\partial^{\nabla^q_\pm} = -\bar\partial^{\nabla^{q+1}_\pm}\partial^{\nabla^q_\pm}$$
$$\Delta^\pm_q = (\bar\partial^{\nabla^q_\pm})^*\bar\partial^{\nabla^q_\pm} = -\partial^{\nabla^{q-1}_\pm}\bar\partial^{\nabla^q_\pm}.$$

In particular, one has

$$\mathcal{D}^-\mathcal{D}^+ = 2\,\Delta^+_0$$
$$\mathcal{D}^+\mathcal{D}^- = 2\,\Delta^0_-.$$

Lemma 3.7. *The Laplacians of the elliptic chains $(\mathcal{C}^\bullet(\mathbb{E}^\pm), \partial^{\nabla^\bullet_\pm}, \bar\partial^{\nabla^\bullet_\pm})$ verify*

$$\Delta^q_\pm - \Delta^\pm_q = -q\frac{\kappa}{2} \mp \frac{\kappa}{4} + B.$$

The following proposition will allow us to compute the holomorphic part of the spectrum of the square of Dirac operator.

Proposition 3.8. *The Laplacians of the elliptic chains $(\mathcal{C}^\bullet(\mathbb{E}^\pm), \partial^{\nabla^\bullet_\pm}, \bar\partial^{\nabla^\bullet_\pm})$ fulfill the following relationships*

$$\Delta^\pm_{-q}\partial^{\nabla^{-(q+1)}_\pm} - \partial^{\nabla^{-(q+1)}_\pm}\Delta^\pm_{-(q+1)} = \left[(q+1)\frac{\kappa}{2} \mp \frac{\kappa}{4} + B\right]\partial^{\nabla^{-(q+1)}_\pm}$$
$$\Delta^q_\pm\bar\partial^{\nabla^{(q+1)}_\pm} - \bar\partial^{\nabla^{(q+1)}_\pm}\Delta^{(q+1)}_\pm = \left[(q+1)\frac{\kappa}{2} \pm \frac{\kappa}{4} - B\right]\bar\partial^{\nabla^{(q+1)}_\pm}.$$

Proof. According to Lemma 3.7 we have

$$\begin{aligned}\Delta^\pm_{-q}\partial^{\nabla^{-(q+1)}_\pm} &= -\partial^{\nabla^{-(q+1)}_\pm}\bar\partial^{\nabla^{-q}_\pm}\partial^{\nabla^{-(q+1)}_\pm} = \partial^{\nabla^{-(q+1)}_\pm}\Delta^{-(q+1)}_\pm \\ &= \partial^{\nabla^{-(q+1)}_\pm}\left[\Delta^\pm_{-(q+1)} + (q+1)\frac{\kappa}{2} \mp \frac{\kappa}{4} + B\right].\end{aligned}$$

The second equality is proved in the same way. $\square$

Remark 3.9. Thanks to the natural identification $K_S^q \otimes \mathbb{E}^- \simeq K_S^{q-1} \otimes \mathbb{E}^+$, the operators $\mathcal{D}_+^q = \sqrt{2}\,\bar{\partial}^{\nabla_+^q}$, $D_-^q = \sqrt{2}\,(\bar{\partial}^{\nabla_+^q})^*$ give morphisms

$$\mathcal{D}_+^q : \Omega^0(S, K_S^q \otimes \mathbb{E}^+) \longrightarrow \Omega^0(S, K_S^q \otimes \mathbb{E}^-)$$
$$\mathcal{D}_-^q : \Omega^0(S, K_S^q \otimes \mathbb{E}^-) \longrightarrow \Omega^0(S, K_S^q \otimes \mathbb{E}^+)$$

which coincide with the graded components of the Dirac operator associated with the twisted Clifford module $\mathbb{S} \otimes (L \otimes K_S^q) = \mathbb{E} \otimes K_S^q$. These operators give chain morphisms of degree zero

$$\mathcal{D}_+^\bullet : \mathcal{C}^\bullet(\mathbb{E}^+) \longrightarrow \mathcal{C}^\bullet(\mathbb{E}^-)$$
$$\mathcal{D}_-^\bullet : \mathcal{C}^\bullet(\mathbb{E}^-) \longrightarrow \mathcal{C}^\bullet(\mathbb{E}^+).$$

4. Relationship between the spectra of the twisted Dirac operator $\mathcal{D}$ and its square $\mathcal{D}^2$

The twisted Dirac operator $\mathcal{D}$ and its square $\mathcal{D}^2$ are elliptic operators and since S is a compact manifold it follows that their spectra are discrete (see for instance [13]). The Dirac operator is positive and formally self-adjoint, therefore the eigenvalues of $\mathcal{D}^2$ are non-negative real numbers and $\mathrm{Ker}(\mathcal{D}) = \mathrm{Ker}(\mathcal{D}^2)$ with $\mathrm{Ker}\,\mathcal{D}^+ = \mathrm{Ker}\,\mathcal{D}^-\mathcal{D}^+$ and $\mathrm{Ker}\,\mathcal{D}^- = \mathrm{Ker}\,\mathcal{D}^+\mathcal{D}^-$.

In what follows we show how the rest of the spectral resolution of $\mathcal{D}$ is determined by the spectral resolution of $\mathcal{D}^2$. This follows immediately from the existence of a natural quaternionic structure on every eigenspace of $\mathcal{D}^2$ corresponding to a non-zero eigenvalue.

Definition 4.1. Let V be a complex vector space. A complex quaternionic structure on V consists of a pair $I,\ J \in \mathrm{End}_{\mathbb{C}}(V)$ verifying $I^2 = J^2 = -\mathrm{Id}_V$ and $IJ = -JI$.

If we denote by $V_I^\pm = \mathrm{Ker}(I \mp i\,\mathrm{Id}_V)$, $V_J^\pm = \mathrm{Ker}(J \mp i\,\mathrm{Id}_V)$, then we have

$$V = V_I^+ \oplus V_I^- = V_J^+ \oplus V_J^-$$

and the projectors onto the factors $V_I^\pm$, $V_J^\pm$ are given by $\pi_I^\pm = \frac{1}{2}(\mathrm{Id}_V \mp i\,I)$, $\pi_J^\pm = \frac{1}{2}(\mathrm{Id}_V \mp i\,J)$, respectively. The following is elementary

Lemma 4.2. *Let $\{I,\ J\}$ be a complex quaternionic structure on V. If $\alpha, \beta \in \{+, -\}$ then*

1. $V_I^\alpha \cap V_J^\beta = 0$.
2. *The induced maps $I \colon V_J^\alpha \to V_J^{-\alpha}$, $J \colon V_I^\alpha \to V_I^{-\alpha}$ are isomorphisms.*
3. *The restrictions of the projectors yield the isomorphisms $\pi_I^\alpha \colon V_J^\beta \to V_I^\alpha$, $\pi_J^\alpha \colon V_I^\beta \to V_J^\alpha$.*

For any $\lambda^2 \in \mathrm{Spec}(\mathcal{D}^2) \setminus \{0\}$, we denote by

$$\mathbb{E}_{\lambda^2}(\mathcal{D}^2) = \{s \in \Omega^0(S, \mathbb{E}) / \mathcal{D}^2 s = \lambda^2 s\}$$

the corresponding space of eigensections.

Proposition 4.3. $\mathbb{E}_{\lambda^2}(\mathcal{D}^2)$ *has a natural complex quaternionic structure defined by* $\left\{I = \frac{i}{\lambda}\,\mathcal{D},\ J = i\,\sigma\right\}$ *where σ is the degree operator of the $\mathbb{Z}_2$-gradation of $\mathbb{E}_{\lambda^2}(\mathcal{D}^2)$ induced by $\mathbb{E} = \mathbb{E}^+ \oplus \mathbb{E}^-$.*

Decomposing $\mathbb{E}_{\lambda^2}(\mathcal{D}^2)$ with respect to I as in Lemma 4.2 we get

$$\mathbb{E}_{\lambda^2}(\mathcal{D}^2) = \mathbb{E}_{-\lambda}(\mathcal{D}) \oplus \mathbb{E}_\lambda(\mathcal{D})$$

where $\mathbb{E}_{\pm\lambda}(\mathcal{D}) = \{s \in \Omega^0(S, \mathbb{E}) / \mathcal{D}s = \pm\lambda s\}$. Therefore $\lambda^2 \in \mathrm{Spec}(\mathcal{D}^2) \setminus \{0\}$ if and only if $\{-\lambda, \lambda\} \subset \mathrm{Spec}(\mathcal{D}) \setminus \{0\}$, and then $\dim \mathbb{E}_{\lambda^2}(\mathcal{D}^2) = 2 \dim \mathbb{E}_{\pm\lambda}(\mathcal{D})$. The projector onto $\mathbb{E}_{\pm\lambda}(\mathcal{D})$ is given by

$$\pi_{\pm\lambda}(s) = \frac{1}{2\lambda}[\lambda s \pm \mathcal{D}(s)].$$

On the other hand, the decomposition with respect to J is given by

$$\mathbb{E}_{\lambda^2}(\mathcal{D}^2) = \mathbb{E}_{\lambda^2}(\mathcal{D}^-\mathcal{D}^+) \oplus \mathbb{E}_{\lambda^2}(\mathcal{D}^+\mathcal{D}^-)$$

where $\mathbb{E}_{\lambda^2}(\mathcal{D}^-\mathcal{D}^+) = \mathbb{E}_{\lambda^2}(\mathcal{D}^2) \cap \Omega^0(S, \mathbb{E}^+)$ and $\mathbb{E}_{\lambda^2}(\mathcal{D}^+\mathcal{D}^-) = \mathbb{E}_{\lambda^2}(\mathcal{D}^2) \cap \Omega^0(S, \mathbb{E}^-)$. Taking into account Lemma 4.2, it follows that

$$\mathcal{D}^+ : \mathbb{E}_{\lambda^2}(\mathcal{D}^-\mathcal{D}^+) \to \mathbb{E}_{\lambda^2}(\mathcal{D}^+\mathcal{D}^-)$$
$$\mathcal{D}^- : \mathbb{E}_{\lambda^2}(\mathcal{D}^+\mathcal{D}^-) \to \mathbb{E}_{\lambda^2}(\mathcal{D}^-\mathcal{D}^+)$$

are isomorphisms. Therefore, $\dim \mathbb{E}_{\lambda^2}(\mathcal{D}^-\mathcal{D}^+) = \dim \mathbb{E}_{\lambda^2}(\mathcal{D}^+\mathcal{D}^-)$ and we have the following equality between spectra

$$\mathrm{Spec}(\mathcal{D}^2) \setminus \{0\} = \mathrm{Spec}(\mathcal{D}^-\mathcal{D}^+) \setminus \{0\} = \mathrm{Spec}(\mathcal{D}^+\mathcal{D}^-) \setminus \{0\}.$$

From these considerations we get the

Proposition 4.4. *The spectral resolution of $\mathcal{D}$ is completely determined by its kernel and the spectral resolution of $\mathcal{D}^-\mathcal{D}^+$ or $\mathcal{D}^+\mathcal{D}^-$. In particular for any $\lambda^2 \in \mathrm{Spec}(\mathcal{D}^2) \setminus \{0\}$ we have the isomorphisms*

$$\pi_{\pm\lambda} : \mathbb{E}_{\lambda^2}(\mathcal{D}^-\mathcal{D}^+) \to \mathbb{E}_{\pm\lambda}(\mathcal{D})$$
$$\pi_{\pm\lambda} : \mathbb{E}_{\lambda^2}(\mathcal{D}^+\mathcal{D}^-) \to \mathbb{E}_{\pm\lambda}(\mathcal{D})$$

defined by $\pi_{\pm\lambda}(s^\pm) = \frac{1}{2\lambda}[\lambda s^\pm \pm \mathcal{D}^\pm(s^\pm)]$ where $s^+ \in \mathbb{E}_{\lambda^2}(\mathcal{D}^-\mathcal{D}^+)$ and $s^- \in \mathbb{E}_{\lambda^2}(\mathcal{D}^+\mathcal{D}^-)$.

Corollary 4.5. *The multiplicity of any $\lambda \in \mathrm{Spec}(\mathcal{D}) \setminus \{0\}$ coincides with the multiplicity of $\lambda^2 \in \mathrm{Spec}(\mathcal{D}^-\mathcal{D}^+) \setminus \{0\}$.*

5. Spectral resolution of the twisted Dirac operator on Riemann surfaces of constant curvature

In what follows we study the spectral resolution of $\mathcal{D}^-\mathcal{D}^+$ on a Riemann surface by means of holomorphic techniques.

Since $\mathbb{E}^+ = K_S^{\frac{1}{2}} \otimes L$ and $\mathbb{E}^- = K_S^{-\frac{1}{2}} \otimes L$, we have $\deg \mathbb{E}^+ = \deg L + \frac{1}{2} \deg K_S$ and $\deg \mathbb{E}^- = \deg L - \frac{1}{2} \deg K_S$. If S has a Riemannian metric g of constant scalar curvature $\kappa \neq 0$ and $L \to S$ is a Hermitian line bundle with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -iB\,\omega \in \Omega^{(1,1)}(S)$, then we have $\deg L = \frac{2B}{\kappa}\chi(S) = -2\frac{B}{\kappa} \deg K_S$, where $\chi(S)$ is the Euler–Poincaré characteristic of S.

5.1. Spectral resolutions of $\mathcal{D}^-\mathcal{D}^+$ and $\mathcal{D}$ on $\mathbb{C}P^1$

We endow $\mathbb{C}P^1$ with the Fubini–Study metric g with constant scalar curvature κ. Notice that for $\kappa = \frac{2}{r^2}$, the Riemannian manifold $(\mathbb{C}P^1, g)$ is isometric to the Euclidean sphere S_r^2 of radius r.

Theorem 5.1. *Let $L \to \mathbb{C}P^1$ be a Hermitian line bundle with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -i\, B\, \omega$, then*

1. *The spectrum of the operator $\mathcal{D}^-\mathcal{D}^+$ on $\mathbb{C}P^1$, for the metric of constant scalar curvature κ, is the set*

$$\mathrm{Spec}(\mathcal{D}^-\mathcal{D}^+) = \left\{ E_q = \frac{\kappa}{2}\{(q+a)^2 + (q+a)\,|\deg L|\}\ \forall q \in \mathbb{Z}, q \geq 0 \right\}$$

where $a = 0$ if $\deg L \geq 1$ and $a = 1$ if $\deg L < 1$.

2. *If $\deg L \geq 1$ then the space of eigensections of $\mathcal{D}^-\mathcal{D}^+$ with eigenvalue E_q gets identified with $H^0(\mathbb{CP}^1, K^{-q}_{\mathbb{C}P^1} \otimes \mathbb{E}^+)$. In the same way, if $\deg L < 1$, then the space of eigensections with eigenvalue E_q gets identified with $H^0(\mathbb{CP}^1, K^{-q}_{\mathbb{C}P^1} \otimes (\mathbb{E}^+)^{-1})$. Therefore the multiplicity of E_q is*

$$m(E_q) = 1 + |\deg L - 1| + 2q.$$

Proof. Let us suppose that $\deg L \geq 1$. Then $\deg \mathbb{E}^+ \geq 0$ and for any $q \geq 0$ the Riemann–Roch theorem gives

$$\dim H^0(\mathbb{CP}^1, K^{-q}_{\mathbb{C}P^1} \otimes \mathbb{E}^+) = 1 + \deg \mathbb{E}^+ + 2q$$

since, by Serre duality $H^1(\mathbb{C}P^1, K^{-q}_{\mathbb{C}P^1} \otimes \mathbb{E}^+) = 0$, due to the fact that $\deg K_{\mathbb{C}P^1} = -2$. Given $0 \neq s^{-q} \in H^0(\mathbb{C}P^1, K^{-q}_{\mathbb{C}P^1} \otimes \mathbb{E}^+)$ we define

$$s_q = \partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} s^{-q} \in \Omega^0(\mathbb{C}P^1, \mathbb{E}^+) \tag{4}$$

which is not zero by Riemann–Roch.

Now if we repeatedly apply the equality

$$\Delta^+_{-q}\partial^{\nabla_+^{-(q+1)}} - \partial^{\nabla_+^{-(q+1)}} \overset{+}{\Delta}_{-(q+1)} = \left[(q+1)\frac{\kappa}{2} - \frac{\kappa}{4} + B\right] \partial^{\nabla_+^{-(q+1)}}$$

proved in Proposition 3.8, we obtain

$$\begin{aligned}
\mathcal{D}^-\mathcal{D}^+ s_q &= 2\,\Delta^+_0 s_q = 2\,\Delta^+_0 \partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} s_q \\
&= 2\left[\partial^{\nabla_+^{-1}} \overset{+}{\Delta}_{-1} \partial^{\nabla_+^{-2}} \ldots \partial^{\nabla_+^{-q}} s^{-q} + \left(\frac{\kappa}{2} - \frac{\kappa}{4} + B\right) s_q\right] \\
&= 2\left[\partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} \Delta^+_{-q} s^{-q} + \left(\frac{q(q+1)}{2}\frac{\kappa}{2} + q\left(-\frac{\kappa}{4} + B\right)\right) s_q\right] \\
&= 2\left[\frac{q(q+1)}{2}\frac{\kappa}{2} - q\frac{\kappa}{4} + qB\right] s_q = \frac{\kappa}{2}\left[q^2 + q\frac{4B}{\kappa}\right] s_q
\end{aligned}$$

since $\Delta^+_{-q} s^{-q} = -\partial^{\nabla_\pm^{q-1}} \bar{\partial}^{\nabla_\pm^q} s^{-q} = 0$ because $s^{-q} \in H^0(\mathbb{C}P^1, K^{-q}_{\mathbb{C}P^1} \otimes \mathbb{E}^+)$ is a holomorphic section. Now taking into account that $\deg L = \frac{4B}{\kappa}$ we get

$$\mathcal{D}^-\mathcal{D}^+ s_q = \frac{\kappa}{2}[q^2 + q \deg L] s_q$$

which proves that

$$E_q = \frac{\kappa}{2}[q^2 + q \deg L]$$

belongs to the spectrum of $\mathcal{D}^-\mathcal{D}^+$. This process can be reversed, therefore the multiplicity of E_q is exactly $\dim H^0(\mathbb{CP}^1, K_{\mathbb{C}P^1}^{-q} \otimes \mathbb{E}^+) = 1 + \deg \mathbb{E}^+ + 2q$. The proof that these are all of the eigenvalues can be found in [27].

In the case where $\deg L < 1$, one has $\deg \mathbb{E}^+ < 0$ and the space of antiholomorphic sections of $K_{\mathbb{C}P^1}^{q} \otimes \mathbb{E}^+$ gets identified in a natural way with $H^0(\mathbb{CP}^1, K_{\mathbb{C}P^1}^{-q} \otimes (\mathbb{E}^+)^{-1})$, and by Riemann–Roch its dimension is $1 + |\deg \mathbb{E}^+| + 2q$. In the same way as in the preceding case, given an antiholomorphic section $0 \neq s^q$ of $K_{\mathbb{C}P^1}^{q} \otimes \mathbb{E}^+$ we get a non-vanishing section of $\mathbb{E}^+$

$$s_q = \bar{\partial}^{\nabla_+^1} \ldots \bar{\partial}^{\nabla_+^q} s^q. \tag{5}$$

By means of Lemma 3.7 for $q = 0$ and in a similar way to the one employed in the preceding case, one proves that

$$\mathcal{D}^-\mathcal{D}^+ s_q = \frac{\kappa}{2}\{(q+1)^2 + (q+1)|\deg L|\} s_q$$

which finishes the proof. □

Remark 5.2. Notice that the formulae (4) and (5) in the proof of Theorem 5.1 provide an explicit way for obtaining the eigensections of $\mathcal{D}^-\mathcal{D}^+$ with eigenvalue E_q in terms of the global holomorphic sections of $\mathcal{O}_{\mathbb{CP}^1}(2q + |\deg L - 1|)$, that is in terms of complex homogeneous polynomials of degree $2q + |\deg L - 1|$ in two variables.

As a consequence we get the

Theorem 5.3. *Let $\mathcal{D}$ be the twisted Dirac operator on $\mathbb{C}P^1$ associated with a metric of constant scalar curvature κ and to a Hermitian line bundle $L \to \mathbb{C}P^1$ with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -i\, B\, \omega$.*

1. *If* $\deg L \geq 1$ *then the non-zero spectrum of $\mathcal{D}$ is the set*

$$\mathrm{Spec}(\mathcal{D}) \setminus \{0\} = \left\{ \pm\sqrt{E_q} = \pm\sqrt{\frac{\kappa}{2}\{q^2 + q\,|\deg L|\}}\ \forall q \in \mathbb{Z}, q \geq 1 \right\}.$$

The space of eigensections with eigenvalue $\pm\sqrt{E_q}$ gets identified with $H^0(\mathbb{CP}^1, K_{\mathbb{C}P^1}^{-q} \otimes \mathbb{E}^+)$ and its multiplicity is

$$m(\pm\sqrt{E_q}) = \deg L + 2q.$$

The kernel of $\mathcal{D}$ is given by $\mathrm{Ker}\,\mathcal{D}^+ = H^0(\mathbb{CP}^1, \mathbb{E}^+)$ *and* $\mathrm{Ker}\,\mathcal{D}^- = 0$, *thus* $\dim \mathrm{Ker}\,\mathcal{D} = \deg L$.

2. *If* $\deg L < 1$ *then the non-zero spectrum of $\mathcal{D}$ is the set*

$$\mathrm{Spec}(\mathcal{D}) \setminus \{0\} = \left\{ \pm\sqrt{E_q} = \pm\sqrt{\frac{\kappa}{2}\{(q+1)^2 + (q+1)|\deg L|\}}\ \forall q \in \mathbb{Z}, q \geq 0 \right\}.$$

The space of eigensections with eigenvalue $\pm\sqrt{E_q}$ gets identified with $H^0(\mathbb{CP}^1, K_{\mathbb{C}P^1}^{-q} \otimes (\mathbb{E}^+)^{-1})$ and its multiplicity is

$$m(\pm\sqrt{E_q}) = 2 + |\deg L| + 2q.$$

The kernel of $\mathcal{D}$ is given by $\mathrm{Ker}\mathcal{D}^+ = 0$ *and* $\mathrm{Ker}\mathcal{D}^- = H^1(\mathbb{CP}^1, \mathbb{E}^+)$, *thus* $\dim \mathrm{Ker}\mathcal{D} = |\deg L|$.

Proof. After Theorem 5.1 and Proposition 4.4 we only have to prove the statements concerning the kernel of $\mathcal{D}$. The Dolbeault resolution and Hodge theory imply $\mathrm{Ker}\mathcal{D}^+ = H^0(\mathbb{CP}^1, \mathbb{E}^+)$ and $\mathrm{Ker}\mathcal{D}^- = H^1(\mathbb{CP}^1, \mathbb{E}^+)$. The proof now follows easily from Riemann–Roch theorem. □

Remark 5.4. For $\kappa = 2$, the spectrum of $\mathcal{D}$ and the multiplicities of the eigenvalues coincide with the results obtained in [6] by a different method. Notice however that our techniques have the additional advantage of giving explicit expressions for the eigensections in terms of the holomorphic sections of the positive spinor bundle twisted by powers of the canonical line bundle.

5.2. *Spectral resolutions of $\mathcal{D}^-\mathcal{D}^+$ and $\mathcal{D}$ on tori*

Let S be a Riemann surface of genus $p = 1$ endowed with a flat Riemannian metric g. It is well known that (S, g) is conformally equivalent to a flat torus $(\mathbb{T}(\tau) = \mathbb{C}/\langle 1, \tau\rangle, g_\tau)$ where g_τ is the flat metric induced by the Euclidean metric of $\mathbb{C}$ and $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$. Moreover, the conformal factor is necessarily constant, therefore (S, g) is isometric to $(\mathbb{T}(\tau), \lambda^2 g_\tau)$ for a suitable $\lambda \in \mathbb{R}^+$.

The arguments in [16, Section 1.4] and [15, Proposition 5.13] for the conformal change of the Dirac operator can be generalized to the twisted case. Let (M, g) be a Riemannian spin manifold and let $\bar{g} = \mathrm{e}^{2u} g$ be a conformal change of the metric. For any spin structure on (M, g) there is a naturally induced spin structure on $(M, \bar{g})$ in such a way that there is an isomorphism $\varphi: \mathbb{S}_g \to \mathbb{S}_{\bar{g}}$ between the corresponding spinor bundles.

Proposition 5.5. *The twisted Dirac operators $\mathcal{D}_g$, $\mathcal{D}_{\bar{g}}$ associated with (M, g), $(M, \bar{g})$, respectively, and to a Hermitian vector bundle E endowed with a unitary connection verify*

$$\mathcal{D}_{\bar{g}}\left(\mathrm{e}^{-\frac{n-1}{2}u}\,\phi(s)\right) = \mathrm{e}^{-\frac{n+1}{2}u}\phi(\mathcal{D}_g\, s)$$

where $\phi: \mathbb{S}_g \otimes E \to \mathbb{S}_{\bar{g}} \otimes E$ is the isomorphism defined by $\varphi \otimes \mathrm{Id}_E$ and $n = \dim M$.

Corollary 5.6. *If the conformal factor $\mathrm{e}^{2u} = \lambda^2$ with $\lambda \in \mathbb{R}^+$ then we have*

$$\mathcal{D}_{\bar{g}}(\phi(s)) = \frac{1}{\lambda}\phi(\mathcal{D}_g\, s).$$

Therefore the spectra are related as $\mathrm{Spec}(\mathcal{D}_{\bar{g}}) = \frac{1}{\lambda}\mathrm{Spec}(\mathcal{D}_g)$, with the same multiplicity for corresponding eigenvalues.

Thus the computation of the spectrum of the twisted Dirac operator $\mathcal{D}$ for a general flat metric g on a Riemann surface S of genus 1 is determined by solving the same problem on a Euclidean flat torus. Therefore, in what follows we assume that (S, g) is isometric to $(\mathbb{T}(\tau), g_\tau)$.

Let $L \to S$ be a Hermitian line bundle with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -iB\,\omega \in \Omega^{(1,1)}(S)$, then its degree is easily seen to be $\deg L = \frac{B}{2\pi}\,\mathrm{Im}\,\tau$. Since $\deg K_S = \deg K_S^{\frac{1}{2}} = 0$, in this case we have $\deg \mathbb{E}^+ = \deg L$.

Now we study the spectral resolution of $\mathcal{D}^-\mathcal{D}^+$ in the case $\deg \mathbb{E}^+ \neq 0$.

Theorem 5.7. *Given a Hermitian line bundle $L \to S$ with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -i\,B\,\omega$ and $\deg L \neq 0$, the spectrum of the operator $\mathcal{D}^-\mathcal{D}^+$ on S, for the flat* *metric g, is the set*

$$\mathrm{Spec}(\mathcal{D}^-\mathcal{D}^+) = \left\{ E_q = 4\pi \frac{|\deg L|}{\operatorname{Im}\tau}(q+a)\ \forall q \in \mathbb{Z}, q \geq 0 \right\}$$

where $a = 0$ *if* $\deg L > 0$ *and* $a = 1$ *if* $\deg L < 0$.

If $\deg L > 0$ *then the space of eigensections with eigenvalue* E_q *gets identified with* $H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$. *In the same way, if* $\deg L < 0$, *then the space of eigensections with eigenvalue* E_q *gets identified with* $H^0(S, K_S^{-q} \otimes (\mathbb{E}^+)^{-1})$. *Therefore the multiplicity of the eigenvalue* E_q *is* $m(E_q) = |\deg L|$.

Proof. If $\deg \mathbb{E}^+ = \deg L > 0$ and $q \geq 0$, by Riemann–Roch we get

$$\dim H^0(S, K_S^{-q} \otimes \mathbb{E}^+) = \deg \mathbb{E}^+$$

since $\deg K_S^{-q} \otimes \mathbb{E}^+ = \deg \mathbb{E}^+$ due to the fact that $K_S = \mathcal{O}_S$.

Given $0 \neq s^{-q} \in H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$ we define

$$s_q = \partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} s^{-q} \in \Omega^0(\mathbb{E}^+).$$

Now we get

$$\Delta^+_{-q}\partial^{\nabla_+^{-(q+1)}} - \partial^{\nabla_+^{-(q+1)}}\Delta^+_{-(q+1)} = B\,\partial^{\nabla_+^{-(q+1)}}$$

and this implies

$$\begin{aligned}
\mathcal{D}^-\mathcal{D}^+ s_q &= 2\,(\bar\partial^{\nabla_{\mathbb{E}^+}})^*\,\bar\partial^{\nabla_{\mathbb{E}^+}} s_q \\
&= 2\,\Delta_0^+ s_q = 2\,\Delta_0^+\partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} s_q \\
&= 2\,[\partial^{\nabla_+^{-1}}\Delta^+_{-1}\partial^{\nabla_+^{-2}} \ldots \partial^{\nabla_+^{-q}} s^{-q} + B\,s_q] \\
&= 2\,[\partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}}\Delta^+_{-q} s^{-q} + qB\,s_q] \\
&= 2qB\,s_q
\end{aligned}$$

since $\Delta^+_{-q}s^{-q} = -\partial^{\nabla_\pm^{q-1}}\bar\partial^{\nabla_\pm^{q}} s^{-q} = 0$ because $s^{-q} \in H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$ is a holomorphic section. Therefore

$$\mathcal{D}^-\mathcal{D}^+ s_q = 2qB\,s_q.$$

Now taking into account that $\deg L = \frac{B}{2\pi}\operatorname{Im}\tau$, we get

$$\mathcal{D}^-\mathcal{D}^+ s_q = 2qB\,s_q = 4\pi\,\frac{q\deg L}{\operatorname{Im}\tau}\,s_q$$

which proves that

$$E_q = 4\pi\,\frac{\deg L}{\operatorname{Im}\,\tau}\,q$$

belongs to the spectrum of $\mathcal{D}^-\mathcal{D}^+$. Since this process can be reversed we obtain the desired multiplicity of E_q In order to see that every eigenvalue is of this form, it is enough to see that the corresponding eigensections generate a dense subspace of $L^2(S, \mathbb{E}^+)$ and this is done by generalization of the method used for the scalar Laplacian or by means of a generalized Selberg trace formula, see [26].

In the case where $\deg \mathbb{E}^+ = \deg L < 0$, the space of antiholomorphic sections of $K_S^q \otimes \mathbb{E}^+$ gets identified in a natural way with $H^0(S, K_S^{-q} \otimes (\mathbb{E}^+)^{-1})$, and by Riemann–Roch its dimension is $|\deg \mathbb{E}^+|$. In the same way as in the preceding case, given an antiholomorphic section $0 \neq s^q$ of $K_S^q \otimes \mathbb{E}^+$ we get a non-vanishing section of $\mathbb{E}^+$

$$s_q = \bar{\partial}^{\nabla_+^1} \ldots \bar{\partial}^{\nabla_+^q} s^q.$$

Following the same procedure as in the preceding case we get

$$\mathcal{D}^- \mathcal{D}^+ s_q = 4\pi \, \frac{|\deg L|}{\operatorname{Im} \tau} (q+1) \, s_q$$

and this finishes the proof. $\square$

Remark 5.8. It is important to point out here the fact that the spectrum of $\mathcal{D}^-\mathcal{D}^+$ does not depend on the chosen line bundle $L \to S$ with $\deg L = k \neq 0$. That is, the spectrum of $\mathcal{D}^-\mathcal{D}^+$ is constant over $\operatorname{Pic}^k(S)$. However, for a given degree of L the spectrum does depend on the point of the moduli space of elliptic curves corresponding to S. It is well known that this moduli space coincides with the quotient of the Siegel upper half-plane by the action of the modular group $\mathbb{P}\mathrm{SL}(2, \mathbb{Z})$. The transformation rule for the eigenvalue $E_q(\tau)$ as a function of the modular parameter τ is

$$E_q(\gamma(\tau)) = |c\tau + d|^2 E_q(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{P}\mathrm{SL}(2, \mathbb{Z})$.

Remark 5.9. If $\deg L > 0$ and we assume the identification $S = \mathbb{T}(\tau)$, then the eigensections of $\mathcal{D}^-\mathcal{D}^+$ with eigenvalue E_q can be naturally identified with the theta functions of the line bundle $K_{\mathbb{T}(\tau)}^q \otimes \mathbb{E}^+$. Taking into account that $\mathbb{T}(\tau)$ is a principally polarized Abelian variety, if the characteristic of the line bundle $K_{\mathbb{T}(\tau)}^q \otimes \mathbb{E}^+$ is $c = a\tau + b$, then the theta functions of this line bundle are expressed in terms of the theta functions $\theta \begin{bmatrix} a \\ b \end{bmatrix}$ with characteristic c, see [21]. One has a similar result for the case $\deg L < 0$.

Theorem 5.10. *Let $\mathcal{D}$ be the twisted Dirac operator on S associated with a flat metric and to a Hermitian line bundle $L \to S$ with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -i\, B\, \omega$.*

1. *If* $\deg L > 0$ *then the non-zero spectrum of $\mathcal{D}$ is the set*

$$\operatorname{Spec}(\mathcal{D}) \setminus \{0\} = \left\{ \pm\sqrt{E_q} = \pm\sqrt{4\pi \, \frac{|\deg L|}{\operatorname{Im} \tau} \{q\}} \; \forall q \in \mathbb{Z}, q \geq 1 \right\}.$$

The space of eigensections with eigenvalue $\pm\sqrt{E_q}$ gets identified with $H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$ and its multiplicity is

$$m(\pm\sqrt{E_q}) = \deg L.$$

The kernel of $\mathcal{D}$ is given by $\operatorname{Ker}\mathcal{D}^+ = H^0(S, \mathbb{E}^+)$ *and* $\operatorname{Ker}\mathcal{D}^- = 0$*, thus* $\dim \operatorname{Ker}\mathcal{D} = \deg L$.

2. *If* $\deg L < 0$ *then the non-zero spectrum of $\mathcal{D}$ is the set*

$$\operatorname{Spec}(\mathcal{D}) \setminus \{0\} = \left\{ \pm\sqrt{E_q} = \pm\sqrt{4\pi \, \frac{|\deg L|}{\operatorname{Im} \tau} \{(q+1)\}} \; \forall q \in \mathbb{Z}, q \geq 0 \right\}.$$

The space of eigensections with eigenvalue $\pm\sqrt{E_q}$ *gets identified with* $H^0(S, K_S^{-q} \otimes (\mathbb{E}^+)^{-1})$ *and its multiplicity is*

$$m(\pm\sqrt{E_q}) = |\deg L|.$$

The kernel of $\mathcal{D}$ *is given by* $\mathrm{Ker}\mathcal{D}^+ = 0$ *and* $\mathrm{Ker}\mathcal{D}^- = H^1(S, \mathbb{E}^+)$, *thus* $\dim \mathrm{Ker}\mathcal{D} = |\deg L|$.

Remark 5.11. Notice that this Theorem implies that on a Riemann surface of genus 1 endowed with a flat metric, the spectrum of the twisted Dirac operator $\mathcal{D}$ associated with a line bundle of non-zero degree does not depend on the spin structure. This is in sharp contrast with the non-twisted case which was studied by Friedrich in [12] and the twisted case for zero degree line bundles as we shall prove below. The latter case is also covered by recent work of Miatello and Podestá on the spectrum of Dirac operators coupled to flat bundles on Bieberbach manifolds, see [24].

Now let us analyze the case $\deg \mathbb{E}^+ = \deg L = 0$. In this case $\mathbb{E}^+$ is a Hermitian line bundle with a flat connection and it has no holomorphic sections unless it is the trivial holomorphic line bundle. Therefore it is necessary to develop a new technique in order to study the spectrum of $\mathcal{D}^-\mathcal{D}^+$ for spinor line bundles of degree zero.

For a complex torus $\mathbb{T} = V/\Lambda$, the Picard group $\mathrm{Pic}^0(\mathbb{T})$ which parametrizes the isomorphism classes of zero degree holomorphic line bundles on $\mathbb{T}$ is given by the dual torus $\mathbb{T}^\vee = V^*/\Lambda^*$ where $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and $\Lambda^* = \{\alpha \in V^* : \alpha(\Lambda) \subset \mathbb{Z}\}$ is the dual lattice. On the other hand $\mathbb{T}^\vee$ can be identified with the group $\mathrm{Hom}(\Lambda, U(1))$ of unitary characters for the lattice Λ by sending $\alpha \in V^*$ to the character $\chi_\alpha = \exp^{2\pi i \alpha(\cdot)}$. The identification $\mathrm{Hom}(\Lambda, U(1)) \simeq \mathrm{Pic}^0(\mathbb{T})$ is defined by sending a character χ to the associated line bundle $L_\chi = V \times_\chi \mathbb{C} \to \mathbb{T}$.

Moreover, the Picard group $\mathrm{Pic}^0(\mathbb{T})$ also parametrizes the gauge equivalence classes of unitary flat connections on the C^∞ trivial line bundle (for instance see [10]). Given $\alpha \in V^*$, the C^∞ line bundle L_{χ_α} is trivialized by the section s defined by the character χ_α. We endow L_{χ_α} with the connection $\nabla_{L_{\chi_\alpha}} \equiv \nabla^{(\alpha)}$ defined by requiring $\nabla^{(\alpha)} s = -2\pi i \alpha \otimes s$.

The canonical line bundle $K_{\mathbb{T}}$ of a complex torus $\mathbb{T}$ is trivial as a holomorphic line bundle. Therefore the holomorphic square roots of $K_{\mathbb{T}}$ correspond exactly to those characters χ such that $\chi^2 = 1$. In terms of the dual torus, they are given by the finite quotient $\left(\frac{1}{2}\Lambda^*\right)/\Lambda^* \subset \mathbb{T}^\vee$.

In our case, $\mathbb{E}^+ = K_S^{\frac{1}{2}} \otimes L$ is completely determined by choosing $\frac{\lambda^*}{2}$, with $\lambda^* \in \Lambda^*$, and $\alpha \in V^*$ in order to describe the flat bundles $K_S^{\frac{1}{2}}$ and L, respectively. Since K_S is trivial as a holomorphic line bundle, the Levi-Civita connection ∇_{K_S} corresponds to the trivial gauge equivalence class and therefore the connection on $K_S^{\frac{1}{2}}$ can be identified with the connection $\nabla^{\left(\frac{\lambda^*}{2}\right)}$ which fulfills $\nabla^{\left(\frac{\lambda^*}{2}\right)} \otimes 1 + 1 \otimes \nabla^{\left(\frac{\lambda^*}{2}\right)} = \nabla_{K_S}$. In the same way the connection ∇_L is gauge equivalent to $\nabla^{(\alpha)}$.

Thus, the Hermitian flat line bundle with connection $(\mathbb{E}^+, \nabla_{\mathbb{E}^+})$ is gauge equivalent to $\left(L_{\chi_{\left(\frac{\lambda^*}{2}+\alpha\right)}}, \nabla^{\left(\frac{\lambda^*}{2}+\alpha\right)}\right)$ and the corresponding spectral problems are unitarily equivalent. In what follows we assume, without loss of generality, this description for the flat bundle $\mathbb{E}^+$. The bundle $\mathbb{E}^+$ has a non-vanishing section $s \in \Omega^0(S, \mathbb{E}^+)$ with trivializes it and such that

$$\nabla_{\mathbb{E}^+} s = -2\pi i \left(\frac{\lambda^*}{2} + \alpha\right) \otimes s$$

where $\frac{\lambda^*}{2} + \alpha$ is understood as a harmonic 1-form on S or equivalently as an invariant 1-form under the translations of the torus S. By means of the flat Riemannian metric g, the invariant forms can be identified with the invariant vector fields. If $A \in \Omega^1(S)$ is a harmonic 1-form, then D_A denotes the invariant vector field associated with it by means of the Riemannian metric g.

Since $\mathbb{E}^+$ is trivial, we can identify $\mathcal{D}^-\mathcal{D}^+$ with a differential operator $\mathcal{P} : C^\infty(S) \to C^\infty(S)$. The explicit expression of this operator is given in the following

Proposition 5.12. *The operator $\mathcal{P} : C^\infty(S) \to C^\infty(S)$ corresponding to $\mathcal{D}^-\mathcal{D}^+$ for the line bundle $\mathbb{E}^+$ by means of a trivializing section s such that $\nabla_{\mathbb{E}^+} s = -2\pi i \left(\frac{\lambda^*}{2} + \alpha\right) \otimes s$ is given by*

$$\mathcal{P} = \Delta + 4\pi\, i D_{\frac{\lambda^*}{2}+\alpha} + 4\pi^2 \left\| \frac{\lambda^*}{2} + \alpha \right\|^2$$

where Δ is the scalar Laplacian of the Riemannian manifold (S, g).

In order to find the spectrum of $\mathcal{P}$ we need the following

Lemma 5.13. *Under the preceding conditions one has*

$$\Delta \circ D_{\frac{\lambda^*}{2}+\alpha} = D_{\frac{\lambda^*}{2}+\alpha} \circ \Delta.$$

Proof. It is enough to prove that the operators commute locally. However, this is evident in any local system of coordinates which is adapted to the covering $\mathbb{C} \to S$. □

Since $\left\| \frac{\lambda^*}{2} + \alpha \right\|^2$ is constant (in a flat torus every harmonic form is parallel), this lemma reduces the computation of the spectral resolution of $\mathcal{P}$ to that of the scalar Laplacian Δ, which is well known since $S = \mathbb{T}(\tau)$ is a torus. For every $\xi \in \Lambda^*$ one defines $f_\xi \in C^\infty(S)$ as

$$f_\xi(z) = \mathrm{e}^{2\pi i \langle \xi, z\rangle}$$

then the spectrum of Δ is

$$\mathrm{Spec}(\Delta) = \{\lambda(\xi) = 4\pi^2 \|\xi\|^2,\ \forall \xi \in \Lambda^*\}$$

and the multiplicity of $\lambda = \lambda(\xi)$ is given by the number of ξ' such that $\lambda(\xi') = \lambda$. The spectral resolution of Δ is formed by the functions $\{f_\xi\}_{\xi\in\Lambda^*}$.

Now we have the following

Theorem 5.14. *If $\deg L = 0$ and $\mathbb{E}^+$ is endowed with the harmonic connection corresponding to $\frac{\lambda^*}{2} + \alpha$, then the spectrum of $\mathcal{D}^-\mathcal{D}^+$ is*

$$\mathrm{Spec}(\mathcal{D}^-\mathcal{D}^+) = \left\{ E_\xi = 4\pi^2 \left\| \xi - \left(\frac{\lambda^*}{2} + \alpha\right) \right\|^2,\ \forall \xi \in \Lambda^* \right\}.$$

The space of eigensections with eigenvalue E_ξ gets identified with the set of functions $f_{\xi'}$ such that $E_{\xi'} = E_\xi$.

Proof. It is enough to take into account that $D_{\frac{\lambda^*}{2}+\alpha} f_\xi = 2\pi i \langle \xi, \frac{\lambda^*}{2} + \alpha\rangle$ and one concludes by means of an easy computation. □

Remark 5.15. We have the following observations regarding Theorem 5.14:

(1) For a line bundle L of degree zero, the spectrum of $\mathcal{D}^-\mathcal{D}^+$ is not constant on $\mathrm{Pic}^0(S)$ and it depends on the spin structure chosen on S.
(2) The formula for the eigenvalues shows that the first eigenvalue of $\mathcal{D}^-\mathcal{D}^+$ is given by the minimal Euclidean length of the points $\xi - \left(\frac{\lambda^*}{2} + \alpha\right)$ where $\xi \in \Lambda^*$.

Let $\mathcal{F}(\Lambda^*)$ be the Dirichlet fundamental domain in the dual space $\mathbb{C}^*$ for the dual lattice Λ^*, see [17]. Let us recall that this is the Voronoi polygon centered around the zero vector in Λ^* (i.e. the set of points which are closer to zero than to any other point of the lattice, see [25]). Therefore, the first eigenvalue of $\mathcal{D}^-\mathcal{D}^+$ is given by the norm of the unique representant in $\mathcal{F}(\Lambda^*)$ for the class determined by $\frac{\lambda^*}{2} + \alpha$ in the dual torus. Since $\mathcal{F}(\Lambda^*)$ can be easily drawn once we have a reduced basis for the lattice Λ^*, we obtain an explicit geometrical and graphical method for determining the lowest eigenvalue of $\mathcal{D}^-\mathcal{D}^+$.

In particular, the zero eigenvalue only appears when $\frac{\lambda^*}{2} + \alpha \in \Lambda^*$. If $\alpha \in \Lambda^*$; that is, L is isomorphic to the trivial holomorphic line bundle, then the zero eigenvalue only appears for the trivial spin structure.

The problem of the multiplicity of the eigenvalues is of arithmetic nature. For the 2-torus $\mathbb{T}(\tau)$, the dual lattice is $\Lambda^* = \left\langle \frac{-i\tau}{\mathrm{Im}\,\tau}, \frac{i}{\mathrm{Im}\,\tau} \right\rangle$. In this basis $\xi \in \Lambda^*$, $\frac{\lambda^*}{2} \in \frac{1}{2}\Lambda^*$ and $\alpha \in \mathbb{C}^*$ have coordinates (n_1, n_2), $\left(\frac{m_1}{2}, \frac{m_2}{2}\right)$ and (a_1, a_2), respectively, with $n_i,\ m_i \in \mathbb{Z}$ and $x_i \in \mathbb{R}$. Then $\left\| \xi - \left(\frac{1}{2}\lambda^* + \alpha\right) \right\|^2$ is given by

$$Q_{\tau,\frac{1}{2}\lambda^*+\alpha}(n_1, n_2) = \frac{1}{\mathrm{Im}\,\tau^2}\{|\tau|^2(n_1 - x_1)^2 - 2\mathbb{R}e\,\tau\,(n_1 - x_1)(n_2 - x_2) + (n_2 - x_2)^2\}$$

where $x_i = \frac{m_i}{2} + a_i$. Therefore the multiplicity of E_ξ is the cardinal of the finite set

$$\mathcal{L}_{\tau,\frac{1}{2}\lambda^*+\alpha}(E_\xi) = \left\{ (n_1, n_2) \in \mathbb{Z}^2 : Q_{\tau,\frac{1}{2}\lambda^*+\alpha}(n_1, n_2) = \frac{E_\xi}{4\pi^2} \right\}.$$

Remark 5.16. It is well known in number theory that $\#\mathcal{L}_{\tau,\frac{1}{2}\lambda^*+\alpha}(E_\xi)$ depends on the arithmetic type of the quadratic form defined by $Q_{\tau,\frac{1}{2}\lambda^*+\alpha}$.

Theorem 5.17. *If* $\deg \mathbb{E}^+ = 0$ *and* $\mathbb{E}^+$ *is endowed with the harmonic connection defined by* $\frac{\lambda^*}{2} + \alpha$, *then the spectrum of the twisted Dirac operator* $\mathcal{D}$ *on* S *is the family*

$$\mathrm{Spec}(\mathcal{D}) = \left\{ \pm\sqrt{E_\xi} = \pm 2\pi \left\| \xi - \left(\frac{\lambda^*}{2} + \alpha\right) \right\|,\ \forall\, \xi \in \Lambda^* \right\}$$

where the spin structure on S *is determined by* $\frac{\lambda^*}{2}$. *The multiplicity of the eigenvalue* $\pm\sqrt{E_\xi}$ *is* $\#\mathcal{L}_{\tau,\frac{1}{2}\lambda^*+\alpha}(E_\xi)$.

Remark 5.18. From the expression for the spectrum one easily sees that the zero eigenvalue only appears in the case $\frac{\lambda^*}{2} + \alpha \in \Lambda^*$. For the non-twisted case (which corresponds to $\alpha \in \Lambda^*$), the zero eigenvalue only appears for the trivial spin structure. This is a well-known fact for the spectrum of the Dirac operator on the 2-torus.

Remark 5.19. Vafa and Witten proved in [30] that there exists a common upper bound for the smallest absolute value of the eigenvalues of all twisted Dirac operators defined on a compact Riemannian spin manifold. This bound depends on the Riemannian metric but it does not depend either on the spin structure or on the Hermitian vector bundle E with connection used.

On a compact Riemann surface, the index theorem implies that a Dirac operator twisted with a Hermitian vector bundle E with $c_1(E) \neq 0$ has non-trivial kernel. Therefore, the Vafa–Witten bound has significant content only when $c_1(E) = 0$. For the flat 2-torus $(\mathbb{T}(\tau) = V/\Lambda, g_\tau)$, the first Vafa–Witten bound can be determined, see [30], using only the family of Dirac operators twisted by all flat line bundles. Then, thanks to Remark 5.15, one can see that

$$\begin{aligned} VW(\mathbb{T}(\tau), g_\tau) &= \max_{\{\lambda^* \in \Lambda^*;\ \alpha \in V^*\}} \min_{\{\xi \in \Lambda^*\}} \left\{ 2\pi \left\| \xi - \left(\frac{\lambda^*}{2} + \alpha \right) \right\| \right\} \\ &= \max \left\{ 2\pi \left\| \frac{\lambda^*}{2} + \alpha \right\| / \forall \frac{\lambda^*}{2} + \alpha \in \mathcal{F}(\Lambda^*) \right\} \\ &= \max \{ 2\pi \|\omega\| / \forall \omega \in \mathcal{F}^\vee(\Lambda^*) \} \end{aligned}$$

where $\mathcal{F}^\vee(\Lambda^*)$ denotes the set of vertices of the Dirichlet fundamental domain of Λ^* (see also [2]).

Taking into account the action of the modular group one can prove that

$$VW(\mathbb{T}(\tau), g_\tau) = \frac{2\pi}{\operatorname{Im}\tau} \max\{\|e\|_{g_\tau} / \forall \xi \in \mathcal{F}^\vee(\Lambda)\}.$$

In particular, if $\{1, \tau\}$ is a reduced basis for Λ with $\mathbb{R}e\, \tau \geq 0$, one has

$$VW(\mathbb{T}(\tau), g_\tau) = \pi \frac{|\tau||\tau - 1|}{(\operatorname{Im}\tau)^2}.$$

5.3. Spectral resolutions of $\mathcal{D}^-\mathcal{D}^+$ and $\mathcal{D}$ on compact Riemann surfaces of genus $p > 1$

Let S be a compact Riemann surface of genus $p > 1$ and let g be the Riemannian metric in the conformal class determined by the complex structure of S which has constant scalar curvature $\kappa = -\frac{2}{r^2}$.

In what follows we study the spectral resolution of $\mathcal{D}^-\mathcal{D}^+$ for a line bundle $\mathbb{E}^+ \to S$ such that $|\deg \mathbb{E}^+| > \deg K_S$. We define the weight $k(\mathcal{L})$ of a line bundle $\mathcal{L} \to S$ as the rational number

$$k(\mathcal{L}) = \frac{\deg \mathcal{L}}{\deg K_S}.$$

Then $k(\mathbb{E}^+) = k(L) + \frac{1}{2}$ and $|\deg \mathbb{E}^+| > \deg K_S$ if and only if $|k(L) + \frac{1}{2}| > 1$, that is $k(L) > \frac{1}{2}$ or $k(L) < -\frac{3}{2}$.

Theorem 5.20. *Let $L \to S$ be a Hermitian line bundle with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -i\, B\, \omega$ such that $|k(L) + \frac{1}{2}| > 1$, then*

1. *The non-empty set* $\operatorname{Spec}^d(\mathcal{D}^-\mathcal{D}^+)$ *defined by*

$$\left\{ E_q = \frac{\kappa}{2}\{(q + a)^2 - 2\,|k(L)|\,(q + a)\},\ \forall q \in \mathbb{Z}, 0 \leq q < \left| k(L) + \frac{1}{2} \right| - 1 \right\}$$

where $a = 0$ if $k(L) \geq \frac{1}{2}$ and $a = 1$ if $k(L) < -\frac{3}{2}$, is included in the spectrum of $\mathcal{D}^-\mathcal{D}^+$ for the metric of constant negative scalar curvature κ on S.

2. *If $k(L) \geq \frac{1}{2}$ then the space of eigensections with eigenvalues E_q gets identified with $H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$. In the same way, if $k(L) < -\frac{3}{2}$, then the space of eigensections with eigenvalues E_q gets identified with $H^0(S, K_S^{-q} \otimes (\mathbb{E}^+)^{-1})$. Therefore the multiplicity of E_q is*

$$m(E_q) = \frac{\chi(S)}{2}\left\{2q - 2\left|k(L) + \frac{1}{2}\right| + 1\right\}.$$

Proof. If $k(L) > \frac{1}{2}$ then one has $\deg \mathbb{E}^+ > \deg K_S$ and the Riemann–Roch theorem implies that

$$\dim H^0(S, K_S^{-q} \otimes \mathbb{E}^+) = \frac{\chi(S)}{2} + \deg \mathbb{E}^+ + q\chi(S)$$

whenever $\deg(K_S^{-q} \otimes \mathbb{E}^+) > \deg K_S$, that is $k(L) - q > \frac{1}{2}$.

Given $s^{-q} \in H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$ one defines $s_q = \partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} s^{-q}$ and we get

$$\begin{aligned}
\mathcal{D}^-\mathcal{D}^+ s_q &= 2\,\Delta_0^+ s_q = 2\,\Delta_0^+ \partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} s_q \\
&= 2\left[\partial^{\nabla_+^{-1}} \Delta_{-1}^+ \partial^{\nabla_+^{-2}} \ldots \partial^{\nabla_+^{-q}} s^{-q} + \left(\frac{\kappa}{2} - \frac{\kappa}{4} + B\right) s_q\right] \\
&= 2\left[\partial^{\nabla_+^{-1}} \ldots \partial^{\nabla_+^{-q}} \Delta_{-q}^+ s^{-q} + \left(\frac{q(q+1)}{2}\frac{\kappa}{2} + q\left(-\frac{\kappa}{4} + B\right)\right) s_q\right] \\
&= 2\left[q^2\frac{\kappa}{4} + qB\right] s_q
\end{aligned}$$

since $\Delta_{-q}^+ s^{-q} = -\partial^{\nabla_+^{-(q+1)}} \bar{\partial}^{\nabla_+^{-q}} s^{-q} = 0$ because $s^{-q} \in H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$ is a holomorphic section. Now taking into account that $k(L) = \frac{\deg L}{\deg K_S} = -2\frac{B}{\kappa}$ we get

$$\mathcal{D}^-\mathcal{D}^+ s_q = \frac{\kappa}{2}[\,q^2 - 2q\,k(L)\,]\,s_q$$

which proves that $E_q = \frac{\kappa}{2}[\,q^2 - 2q\,k(L)\,]$ belongs to the spectrum of $\mathcal{D}^-\mathcal{D}^+$. This process can be reversed, therefore the multiplicity of E_q coincides with $\dim H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$.

In the case where $k(L) < -\frac{3}{2}$ the space of antiholomorphic sections of $K_S^q \otimes \mathbb{E}^+$ gets identified in a natural way with $H^0(S, K_S^{-q} \otimes (\mathbb{E}^+)^{-1})$ which, by Riemann–Roch, has dimension $\frac{\chi(S)}{2} + |\deg \mathbb{E}^+| + q\chi(S)$ whenever $|k(\mathbb{E}^+)| - 1 > 0$. In the same way as in the preceding case, given an antiholomorphic section $0 \neq s^q$ of $K_S^q \otimes \mathbb{E}^+$ one defines

$$s_q = \bar{\partial}^{\nabla_+^1} \ldots \bar{\partial}^{\nabla_+^q} s^q.$$

And one can see that, in this case,

$$\mathcal{D}^-\mathcal{D}^+ s_q = \frac{\kappa}{2}[(q+1)^2 + 2\,k(L)\,(q+1)]s_q$$

which finishes the proof. □

Remark 5.21. The conditions that were imposed on the degree of the line bundle in the preceding theorem guarantee the vanishing of the first cohomology group and therefore we can calculate explicitly the dimension of the space of holomorphic sections. However, those conditions are by no means necessary in order to have non-zero holomorphic sections. Therefore, if one can assure the existence of holomorphic sections by other means, then one can impose weaker conditions in the theorems.

Theorem 5.22. *Let $\mathcal{D}$ be the twisted Dirac operator on S associated with a metric of constant negative scalar curvature κ and with a Hermitian line bundle $L \to S$ with a unitary harmonic connection ∇_L of curvature $F^{\nabla_L} = -i\, B\, \omega$ with $|k(L) + \frac{1}{2}| > 1$.*

1. *If $k(L) > \frac{1}{2}$ then the set*
$$\left\{\pm\sqrt{E_q} = \pm\sqrt{\frac{\kappa}{2}\{q^2 - 2\,k(L)\,q\}},\ \forall q \in \mathbb{Z}, 1 \le q < k(L) - \frac{1}{2}\right\}$$
is included in the non-zero spectrum of $\mathcal{D}$. The space of eigensections with eigenvalue $\pm\sqrt{E_q}$ gets identified with $H^0(S, K_S^{-q} \otimes \mathbb{E}^+)$ and its multiplicity is
$$m(\pm\sqrt{E_q}) = \chi(S)\{q - k(L)\}.$$
The kernel of $\mathcal{D}$ is given by $\mathrm{Ker}\mathcal{D}^+ = H^0(S, \mathbb{E}^+)$ *and* $\mathrm{Ker}\mathcal{D}^- = 0$*, thus* $\dim \mathrm{Ker}\mathcal{D} = \deg\, L$.
2. *If $k(L) < -\frac{3}{2}$ then the set*
$$\left\{\pm\sqrt{E_q} = \pm\sqrt{\frac{\kappa}{2}\{(q+1)^2 + 2\,k(L)\,(q+1)\}},\ \forall q \in \mathbb{Z}, 0 \le q < \left|k(L) + \frac{3}{2}\right|\right\}$$
is included in the non-zero spectrum of $\mathcal{D}$. The space of eigensections with eigenvalue $\pm\sqrt{E_q}$ gets identified with $H^0(S, K_S^{-q} \otimes (\mathbb{E}^+)^{-1})$ and its multiplicity is
$$m(\pm\sqrt{E_q}) = \chi(S)\{q - |k(L)| + 1\}.$$
The kernel of $\mathcal{D}$ is given by $\mathrm{Ker}\mathcal{D}^+ = 0$ *and* $\mathrm{Ker}\mathcal{D}^- = H^1(S, \mathbb{E}^+)$*, thus* $\dim \mathrm{Ker}\mathcal{D} = |\deg\, L|$.

Proof. After Theorem 5.20 and Proposition 4.4, the proof follows from an argument similar to the proof of Theorem 5.3. □

Definition 5.23. The subset of the spectrum given in the previous theorem is called the holomorphic part of the spectrum of $\mathcal{D}$ and it is denoted by $\mathrm{Spec}_{hol}(\mathcal{D})$.

Remark 5.24. The untwisted case or the twisted case for $\deg L = 0$ corresponds to $k(L) = 0$ and therefore they are not covered by Theorem 5.22. However, the well-known results concerning the existence of harmonic spinors on the Riemann surface of genus $p > 1$, and their general dependence on the spin structure and the metric [16,5,8], lead us to conjecture that for $-\frac{3}{2} \le k(L) \le \frac{1}{2}$, the spectrum of the twisted Dirac operator on Riemann surfaces of genus $p > 1$ with metrics of constant curvature depends explicitly on the spin structure.

By the uniformization theorem, any compact Riemann surface S of genus $p > 1$ can be obtained as $S = \mathbb{H}/\Gamma$ where $\mathbb{H}$ is the Poincaré upper half-plane and $\Gamma \subset Sl(2, \mathbb{R})$ is a discrete Fuchsian group. Let us denote by $\pi : \mathbb{H} \to S = \mathbb{H}/\Gamma$ the universal Riemannian covering map.

The group of holomorphic line bundles on S is canonically isomorphic to the group $H^1(S, \mathcal{O}_S^*)$ and one has the exact sequence [21]
$$0 \to H^1(\Gamma, H^0(\mathbb{H}, \mathcal{O}_{\mathbb{H}}^*)) \to H^1(S, \mathcal{O}_S^*) \xrightarrow{\pi^*} H^1(\mathbb{H}, \mathcal{O}_{\mathbb{H}}^*).$$

Therefore non-equivalent holomorphic lines bundles $\mathcal{L}$ on S can be constructed as the quotient of the trivial line bundle on $\mathbb{H}$ by the action of Γ defined by a holomorphic automorphic factor $j : \Gamma \times \mathbb{H} \to \mathbb{C}^*$ (that is, $j(\gamma, -)$ is a holomorphic map fulfilling the cocycle condition $j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2(z))\, j(\gamma_2, z)$). The space of sections $\Omega^0(S, \mathcal{L}) = \Omega^0(\mathbb{H}, \mathbb{H} \times \mathbb{C})^{\Gamma} = \{s \in H^0(\mathbb{H}, \mathcal{O}^*_{\mathbb{H}})/s(\gamma(z)) = j(\gamma, z)\, s(z)\}$.

Twisted-spinor line bundles $\mathbb{E}^{\pm}$ on S can be constructed in this way taking the automorphic factor

$$j(\gamma, z) = \mu(\gamma)\, \mathrm{e}^{i\, 2\, k(\mathbb{E}^{\pm})\, arg(c_\gamma z + d_\gamma)}$$

where $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma$ and $\mu : \Gamma \to U(1)$ is a Γ-multiplier system of weight $k(\mathbb{E}^{\pm})$. The existence of such multipliers of weight $k(\mathbb{E}^{\pm})$ is guaranteed since $k(\mathbb{E}^{\pm})\chi(S) \in \mathbb{Z}$ [14]. Therefore the twisted spinor fields on S can be thought as Γ-automorphic functions on the upper half-plane of weight $k(\mathbb{E}^{\pm})$ and multiplier μ; that is, functions $f^{\pm} : \mathbb{H} \to \mathbb{C}$ fulfilling

$$f^{\pm}(\gamma(z)) = \mu(\gamma)\, \mathrm{e}^{i 2\, k(\mathbb{E}^{\pm})\, arg(c_\gamma z + d_\gamma)}\, f^{\pm}(z) \qquad \forall \gamma \in \Gamma.$$

These facts explain the definition of the weight of a line bundle given in our discussion of the spectrum of the twisted Dirac operator.

With these identifications, the twisted Dirac operators $\mathcal{D}^+$ and $\mathcal{D}^-$ coincide with the raising and lowering Maass operators of weight $k(\mathbb{E}^{\pm})$ [11]

$$R_{k(\mathbb{E}^{\pm})} = -i \left[(z - \bar{z}) \frac{\partial}{\partial z} - k(\mathbb{E}^{\pm}) \right]$$
$$L_{k(\mathbb{E}^{\pm})} = -i \left[(z - \bar{z}) \frac{\partial}{\partial \bar{z}} + k(\mathbb{E}^{\pm}) \right]$$

and the operator $\mathcal{D}^2$ is determined by the Maass Laplacians

$$\Delta_{k(\mathbb{E}^{\pm})} = -(z - \bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} - k(\mathbb{E}^{\pm})\, (z - \bar{z}) \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right).$$

The complete spectral resolution of $\mathcal{D}^2$ on a compact Riemann surface for genus $p > 1$ is unknown in general. However the construction of the spectral resolution, as we have shown above, is equivalent to finding the explicit expressions of the Maass automorphic forms. Once this connection is established, we can make use of the information provided by all the particular cases in which it is possible to determine the spaces of Maass automorphic forms, see [23,26,28] for more details.

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